5.2 The intuitive (but non-converging) L-Curve Criterion

Take a closer look into the regularized solution, combine Tikhonov solution and the TSVD solution together. First of all, we define filter matrix as follows:

where the filter factors for Tikhonov, and for TSVD. Thus we can write the regularized solution as:

Remind that the right-hand side is the exact signal plus additive noise , i.e. , then compute the relative error , we have

Where is called the regularization error and is called the perturbation error.

Analysis for the Tikhonov method:

If  is very small then , hence and is small but the perturbation error will be large because we apply very little filtering. On the other hand, if is very large then many filter factors will be small and thus the perturbation error will be small, but then is not close to the identity matrix , so the regularization error will be large.

The goal is to choose a  is to balance the size of two error terms and .

**L-curve criterion: have two distinctly different parts**

Parr I: flat, because of regularization error dominates

Part II: vertical, because of perturbation error dominates

Between them is the transition region which is associated with those values of or for which the dominating error component changes between the perturbation error and the regularization error .

Our goal is to choose a value that corresponds to the L-curve’s corner, in the hope that this value will provide a good balance of the perturbation error and the regularization error.

By the supported analysis above and an intuitive idea, we choose a value of that corresponds to the L-curve’s corner, where the curvature of the curve is the largest. We continue to measure this in log-log system.

It’s much more convenient to consider the L-curve for Tikhonov regularization here. We introduce the quantities as follows:

and

such that the L-curve is given by . If denotes differentiation with respect to , then it follows that and . When we use these relations, then it follows immediately that the second derivatives of and are given by

and

When we insert these relations to the definition of the curvature

where and .

Conclusion: choose  such that the curvature is maximum.

While for TSVD, the discussion above is not valid because the set of solutions is finite. But it still often makes good sense to locate the TSVD solution at the overall corner of the discrete L-curve:

Conclusion: choose at the overall corner of the discrete L-curve.

While how to actually compute this corner is not as straightforward as it may seem, because a discrete L-curve can have many small local corners. One can refer to [**HaJeRo**] for a recent algorithm.

Exceptions:

When the SVD components of the exact solution decay fast to zero, the L-curve’s corner is typically at the point where the solution norm starts to increase dramatically, but for very smooth solutions this happens when we have included too many noisy components, and hence will lead to an under-regularized solution.

If we let the noise go to zero, then it has been shown that the regularization parameter tends to diverge from the optimal one. Fortunately, there is a rare situation in practise.